An Inner Product on $C[a,b]$

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Let $C[a,b]$ be the set of all continuous functions on the closed interval $[a,b]$. It is not difficult to show that $C[a,b]$ is a vector space over $\mathbb{R}$. Let f and g be vectors in $C[a,b]$ and define

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx. \tag{1}$$

We claim that $\langle\ \ ,\ \ \rangle$ is an innerproduct defined on $C[a,b]$.

In order to prove our claim, we need to show four things:

1. $\langle f, f\rangle \geq 0$ with equality if and only if $f = 0$,
2. $\langle f, g\rangle = \langle g, f\rangle$,
3. $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$, and finally
4. $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$.

For example, I offer of proof of the first part of (1). Let $f \in C[a,b]$. Then, because $[f(x)]^2 \geq 0$ on $[a,b]$, the graph of f lies either on or above the x-axis on the interval $[a,b]$. Hence, the area under the graph of f on $[a,b]$ is nonnegative. That is,

$$\langle f, f\rangle = \int_a^b f(x)f(x)\,dx = \int_a^b [f(x)]^2\,dx \geq 0.$$

As another example, I will prove (2). Let $f, g \in C[a,b]$. Then, because $f(x)$ and $g(x)$ are real numbers for all $x \in [a,b]$, they commute; i.e., $f(x)g(x) = g(x)f(x)$ for all $x \in [a,b]$. Thus, we can write

$$\begin{aligned}
\langle f, g\rangle &= \int_a^b f(x)g(x)\,dx \\
&= \int_a^b g(x)f(x)\,dx \\
&= \langle g, f\rangle.
\end{aligned}$$

The other properties are proved similarly. The only tricky thing about this extra credit assignment is the second part of part (1). That will require an appeal to continuity.